Problem Set V, Selected Solutions

Patrick Ryan

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Problem 0.1. Suppose $T \in \mathcal{L}(V)$. Prove that the intersection of any collection of subspaces of V invariant under T is invariant under T.

Proof. Consider a collection $\{U_i\}_{i \in I}$ of subspaces of V invariant under T where I is an arbitrary indexing set. Suppose that

$$u \in \bigcap_{i \in I} U_i. \tag{0.1}$$

Then $u \in U_i$ for all $i \in I$. Thus, $Tu \in U_i$ for all $i \in I$ by invariance. Hence, $Tu \in \bigcap_{i \in I} U_i$ and our intersection is invariant under T.

Remark 0.2. Many of you proved this for only two spaces or a finite indexing set. Note that the problem says an "intersection of *any* collection." Pay attention to such things!

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Problem 0.3. Let V be a finite dimensional vector space over F.

1. Show that an invertible linear operator $T : F^2 \to F^2$ maps a basis to another basis.
2. Suppose that $F = \mathbb{Z}/2\mathbb{Z}$. What is the probability that a linear operator $T : F^2 \to F^2$ is not invertible when randomly choosing out of all such maps?

Proof.

1. Suppose that $T : F^2 \to F^2$ is an invertible linear operator. Let (v_1, v_2) be a basis of F^2. We prove that (Tv_1, Tv_2) is also a basis of F^2. We write any $v \in F^2$ as

$$v = \alpha_1 v_1 + \alpha_2 v_2, \tag{0.2}$$

for $\alpha_1, \alpha_2 \in F$. Applying T

$$Tv = T(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 T v_1 + \alpha_2 T v_2 \tag{0.3}$$

by linearity. We know that T is surjective as it is invertible, hence (Tv_1, Tv_2) spans F^2. Finally, since $\dim(Tv_1, Tv_2) = \dim F^2$, we have that (Tv_1, Tv_2) is a basis of F^2.

2. Since $F = \mathbb{Z}/2\mathbb{Z}$ its only elements are $1, 0$. We take $\{(1,0), (0,1)\}$ as a basis of F^2. Our operator T is given by a general matrix

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}. \tag{0.4}$$

We see that there are 16 possible ways to write our matrix. We require that T is invertible. This forces three choices for the first column of our matrix:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{0.5}$$

Choosing, $T(v_1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ yields $T(v_2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ or $T(v_2) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Similarly, the other two options for our first column give two choices for the second column. Thus, there are six invertible maps. Finally, the probability of choosing a non-invertible map is $5/8$.

$\square$